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*Research Article*

# Existence of Solutions for Nonlinear Four-Point $p$-Laplacian Boundary Value Problems on Time Scales

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We are concerned with proving the existence of positive solutions of a nonlinear second-order four-point boundary value problem with a $p$-Laplacian operator on time scales. The proofs are based on the fixed point theorems concerning cones in a Banach space. Existence result for $p$-Laplacian boundary value problem is also given by the monotone method.



## 1. Introduction

Let $\mathbb{T}$ be any time scale such that $[0,1]$ be subset of $\mathbb{T}$. The concept of dynamic equations on time scales can build bridges between differential and difference equations. This concept not only gives us unified approach to study the boundary value problems on discrete intervals with uniform step size and real intervals but also gives an extended approach to study on discrete case with non uniform step size or combination of real and discrete intervals. Some basic definitions and theorems on time scales can be found in [1, 2].

In this paper, we study the existence of positive solutions for the following nonlinear four-point boundary value problem with a $p$-Laplacian operator:

$$\left(\phi_p\left(x^{\Delta}\right)\right)^{\nabla}(t) + h(t)f(t,x(t)) = 0, \quad t \in [0,1], \tag{1.1}$$

$$\alpha\phi_p(x(\rho(0)) - \Psi\left(\phi_p\left(x^{\Delta}(\xi)\right) = 0, \qquad \gamma\phi_p(x(\sigma(1))) + \delta\phi_p(x^{\Delta}(\eta) = 0, \tag{1.2}$$

where $\phi_p(s)$ is an operator, that is, $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$, where $1/p + 1/q = 1$, $\alpha, \gamma > 0$, $\delta \geq 0$, $\xi, \eta \in (\rho(0), \sigma(1))$ with $\xi < \eta$:

(H1) the function $f \in C([0,1] \times [0,\infty), [0,\infty))$,

(H2) the function $h \in C_{ld}(\mathbb{T}, [0,\infty))$ and does not vanish identically on any closed subinterval of $[\rho(0), \sigma(1)]$ and $0 < \int_{\rho(0)}^{\sigma(1)} h(t)\nabla t < \infty$,

(H3) $\Psi : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies that there exist $B_2 \geq B_1 > 0$ such that $B_1 s \leq \Psi(s) \leq B_2 s$, for $s \in [0,\infty)$.

In recent years, the existence of positive solutions for nonlinear boundary value problems with $p$-Laplacians has received wide attention, since it has led to several important mathematical and physical applications [3, 4]. In particular, for $p = 2$ or $\phi_p(s) = s$ is linear, the existence of positive solutions for nonlinear singular boundary value problems has been obtained [5, 6]. $p$-Laplacian problems with two-, three-, and $m$-point boundary conditions for ordinary differential equations and difference equations have been studied in [7–9] and the references therein. Recently, there is much attention paid to question of positive solutions of boundary value problems for second-order dynamic equations on time scales, see [10–13]. In particular, we would like to mention some results of Agarwal and O'Regan [14], Chyan and Henderson [5], Song and Weng [15], Sun and Li [16], and Liu [17], which motivate us to consider the $p$-Laplacian boundary value problem on time scales.

The aim of this paper is to establish some simple criterions for the existence of positive solutions of the $p$-Laplacian BVP (1.1)-(1.2). This paper is organized as follows. In Section 2 we first present the solution and some properties of the solution of the linear $p$-Laplacian BVP corresponding to (1.1)-(1.2). Consequently we define the Banach space, cone and the integral operator to prove the existence of the solution of (1.1)-(1.2). In Section 3, we state the fixed point theorems in order to prove the main results and we get the existence of at least one and two positive solutions for nonlinear $p$-Laplacian BVP (1.1)-(1.2). Finally, using the monotone method, we prove the existence of solutions for $p$-Laplacian BVP in Section 4.

## 2. Preliminaries and Lemmas

In this section, we will give several fixed point theorems to prove existence of positive solutions of nonlinear $p$-Laplacian BVP (1.1)-(1.2). Also, to state the main results in this paper, we employ the following lemmas. These lemmas are based on the linear dynamic equation:

$$\left(\phi_p\left(x^{\Delta}\right)\right)^{\nabla}(t) + y(t) = 0. \tag{2.1}$$

**Lemma 2.1.** *Suppose condition (H2) holds, then there exists a constant $\theta \in (\rho(0), (\sigma(1) - \rho(0))/2)$ that satisfies*

$$0 < \int_{\theta}^{\sigma(1)-\theta} h(t)\nabla t < \infty. \tag{2.2}$$

*Furthermore, the function*

$$A(t) = \int_{\theta}^{t} \phi_q\left(\int_{s}^{t} h(u)\nabla u\right)\Delta s + \int_{t}^{\sigma(1)-\theta} \phi_q\left(\int_{t}^{s} h(u)\nabla u\right)\Delta s, \quad t \in [\theta, \sigma(1) - \theta] \tag{2.3}$$

*is a positive continuous function, therefore, $A(t)$ has a minimum on $[\theta,\sigma(1)-\theta]$, hence one supposes that there exists $L>0$ such that $A(t)\geq L$ for $t\in[\theta,\sigma(1)-\theta]$.*

*Proof.* It is easily seen that $A(t)$ is continuous on $[\theta,\sigma(1)-\theta]$.

Let

$$A_1(t)=\int_\theta^t \phi_q\left(\int_s^t h(u)\nabla u\right)\Delta s,\qquad A_2(t)=\int_t^{\sigma(1)-\theta}\phi_q\left(\int_t^s h(u)\nabla u\right)\Delta s. \tag{2.4}$$

Then, from condition (H2), we have that the function $A_1(t)$ is strictly monoton nondecreasing on $[\theta,\sigma(1)-\theta]$ and $A_1(\theta)=0$, the function $A_2(t)$ is strictly monoton nonincreasing on $[\theta,\sigma(1)-\theta]$ and $A_2(\sigma(1)-\theta)=0$, which implies $L=\min_{t\in[\theta,\sigma(1)-\theta]}A(t)>0$. □

Throughout this paper, let $E=C[0,1]$, then $E$ is a Banach space with the norm $\|x\|=\sup_{t\in[0,1]}|x(t)|$. Let

$$K=\{x\in E: x(t)\geq 0,\ x(t)\ \text{concave function on}\ [0,1]\}. \tag{2.5}$$

**Lemma 2.2.** *Let $x(t)\in K$ and $\theta$ be as in Lemma 2.1, then*

$$x(t)\geq\frac{\theta}{\sigma(1)-\rho(0)}\|x\|,\quad \forall t\in[\theta,\sigma(1)-\theta]. \tag{2.6}$$

*Proof.* Suppose $\tau=\inf\{\varsigma\in[\rho(0),\sigma(1)]:\sup_{t\in[\rho(0),\sigma(1)]}x(t)=x(\varsigma)\}$. We have three different cases.

(i) $\tau\in[\rho(0),\theta]$. It follows from the concavity of $x(t)$ that each point on the chard between $(\tau,x(\tau))$ and $(\sigma(1),x(\sigma(1)))$ is below the graph of $x(t)$, thus

$$x(t)\geq x(\tau)+\frac{x(\sigma(1))-x(\tau)}{\sigma(1)-\tau}(t-\tau),\quad t\in[\theta,\sigma(1)-\theta], \tag{2.7}$$

then

$$\begin{aligned} x(t)&\geq\min_{t\in[\theta,\sigma(1)-\theta]}\left[x(\tau)+\frac{x(\sigma(1))-x(\tau)}{\sigma(1)-\tau}(t-\tau)\right]\\ &=x(\tau)+\frac{x(\sigma(1))-x(\tau)}{\sigma(1)-\tau}(\sigma(1)-\theta-\tau)\\ &=\frac{\sigma(1)-\theta-\tau}{\sigma(1)-\tau}x(\sigma(1))+\frac{\theta}{\sigma(1)-\tau}x(\tau)\\ &\geq\frac{\theta}{\sigma(1)-\rho(0)}x(\tau), \end{aligned} \tag{2.8}$$

this means $x(t)\geq(\theta/(\sigma(1)-\rho(0)))\|x\|$ for $t\in[\theta,\sigma(1)-\theta]$.

(ii) $\tau \in [\theta, \sigma(1)-\theta]$. If $t \in [\theta, \tau]$, similarly, we have

$$\begin{aligned}
x(t) &\geq x(\tau) + \frac{x(\tau) - x(\rho(0))}{\tau - \rho(0)}(t-\tau) \\
&\geq x(\tau) + \frac{x(\tau) - x(\rho(0))}{\tau - \rho(0)}(\theta-\tau) \\
&= \frac{\theta - \rho(0)}{\tau - \rho(0)} x(\tau) + \frac{\tau - \theta}{\tau - \rho(0)} x(\rho(0)) \\
&\geq \frac{\theta - \rho(0)}{\sigma(1) - \rho(0)} x(\tau) \geq \frac{\theta}{\sigma(1) - \rho(0)} x(\tau).
\end{aligned} \tag{2.9}$$

If $t \in [\tau, \sigma(1)-\theta]$, similarly, we have

$$\begin{aligned}
x(t) &\geq x(\tau) + \frac{x(\sigma(1)) - x(\tau)}{\sigma(1) - \tau}(t-\tau) \\
&\geq \min_{t\in[\theta,\sigma(1)-\theta]} \left[ x(\tau) + \frac{x(\sigma(1)) - x(\tau)}{\sigma(1) - \tau}(t-\tau) \right] \\
&= \frac{\theta}{\sigma(1) - \tau} x(\tau) + \frac{\sigma(1) - \tau - \theta}{\sigma(1) - \tau} x(\sigma(1)) \\
&\geq \frac{\theta}{\sigma(1) - \rho(0)} x(\tau),
\end{aligned} \tag{2.10}$$

this means $x(t) \geq (\theta/(\sigma(1)-\rho(0)))\|x\|$ for $t \in [\theta, \sigma(1)-\theta]$.

(iii) $\tau \in [\sigma(1)-\theta, \sigma(1)]$. Similarly we have

$$x(t) \geq x(\tau) + \frac{x(\tau) - x(\rho(0))}{\tau - \rho(0)}(t-\tau), \quad t \in [\theta, \sigma(1)-\theta], \tag{2.11}$$

then

$$\begin{aligned}
x(t) &\geq \min_{t\in[\theta,\sigma(1)-\theta]} \left[ x(\tau) + \frac{x(\tau) - x(\rho(0))}{\tau - \rho(0)}(t-\tau) \right] \\
&= \frac{\theta - \rho(0)}{\tau - \rho(0)} x(\tau) + \frac{\tau - \theta}{\tau - \rho(0)} x(\rho(0)) \\
&\geq \frac{\theta}{\sigma(1) - \rho(0)} x(\tau),
\end{aligned} \tag{2.12}$$

this means $x(t) \geq (\theta/(\sigma(1)-\rho(0)))\|x\|$ for $t \in [\theta, \sigma(1)-\theta]$. From the above, we know

$$x(t) \geq \frac{\theta}{\sigma(1)-\rho(0)}\|x\|, \quad t \in [\theta, \sigma(1)-\theta]. \tag{2.13}$$

□

**Lemma 2.3.** *Suppose that condition (H3) holds. Let $y \in C[\rho(0), \sigma(1)]$ and $y(t) \geq 0$. Then p-Laplacian BVP* (2.1)-(1.2) *has a solution*

$$x(t) = \begin{cases} \phi_q\left(\frac{1}{\alpha}\Psi\left(\int_\xi^\tau y(r)\nabla r\right)\right) + \int_{\rho(0)}^t \phi_q\left(\int_s^\tau y(r)\nabla r\right)\Delta s, & \rho(0) \leq t \leq \tau; \\ \phi_q\left(\frac{\delta}{\gamma}\int_\tau^\eta y(r)\nabla r\right) + \int_t^{\sigma(1)} \phi_q\left(\int_\tau^s y(r)\nabla r\right)\Delta s, & \tau \leq t \leq \sigma(1), \end{cases} \tag{2.14}$$

*where $\tau$ is a solution of the following equation*

$$V_1(t) = V_2(t), \quad t \in \left[\rho(0), \sigma(1)\right], \tag{2.15}$$

*where*

$$\begin{aligned} V_1(t) &= \phi_q\left(\frac{1}{\alpha}\Psi\left(\int_\xi^t y(r)\nabla r\right)\right) + \int_{\rho(0)}^t \phi_q\left(\int_s^t y(r)\nabla r\right)\Delta s, \\ V_2(t) &= \phi_q\left(\frac{\delta}{\gamma}\int_t^\eta y(r)\nabla r\right) + \int_t^{\sigma(1)} \phi_q\left(\int_t^s y(r)\nabla r\right)\Delta s. \end{aligned} \tag{2.16}$$

*Proof.* Obviously $V_1(\rho(0)) < 0$ and $V_1(\sigma(1)) > 0$, beside these $V_2(\rho(0)) > 0$ and $V_2(\sigma(1)) < 0$. So, there must be an intersection point between $\rho(0)$ and $\sigma(1)$ for $V_1(t)$ and $V_2(t)$, which is a solution $V_1(t) - V_2(t) = 0$, since $V_1(t)$ and $V_2(t)$ are continuous. It is easy to verify that $x(t)$ is a solution of (2.1)-(1.2). If (2.1) has a solution, denoted by $x$, then $(\phi(x^\Delta))^\nabla(t) = -y(t) \leq 0$. There exists a constant $\tau \in (\rho(0), \sigma(1))$ such that $x^\Delta(\tau) = 0$. If it does not hold, without loss of generality, one supposes that $x^\Delta(t) > 0$ for $(\rho(0), \sigma(1))$. From the boundary conditions, we have

$$\begin{aligned} \phi_p(x(\rho(0))) &= \frac{1}{\alpha}\Psi\left(\phi_p\left(x^\Delta(\xi)\right)\right) > 0, \\ \phi_p(x(\sigma(1))) &= -\frac{\delta}{\gamma}\left(\phi_p\left(x^\Delta(\eta)\right)\right) < 0, \end{aligned} \tag{2.17}$$

which is a contradiction.

Integrating (2.1) on $(\tau, t)$, we get

$$\phi_p\left(x^{\Delta}(t)\right) = -\int_{\tau}^{t} y(s)\nabla s. \tag{2.18}$$

Then, we have

$$\begin{aligned} x^{\Delta}(t) &= \phi_q\left(-\int_{\tau}^{t} y(s)\nabla s\right) = -\phi_q\left(\int_{\tau}^{t} y(s)\nabla s\right), \\ x(t) &= x(\tau) - \int_{\tau}^{t} \phi_q\left(\int_{\tau}^{s} y(r)\nabla r\right)\Delta s. \end{aligned} \tag{2.19}$$

Using the second boundary condition and the formula (2.18) for $t = \eta$, we have

$$x(\sigma(1)) = \phi_q\left(\frac{\delta}{\gamma}\int_{\tau}^{\eta} y(s)\nabla s\right). \tag{2.20}$$

Also, using the formula (2.18), we have

$$\begin{aligned} x(t) &= \phi_q\left(\frac{\delta}{\gamma}\int_{\tau}^{\eta} y(s)\nabla s\right) + \int_{\tau}^{\sigma(1)} \phi_q\left(\int_{\tau}^{s} y(r)\nabla r\right)\Delta s - \int_{\tau}^{t} \phi_q\left(\int_{\tau}^{s} y(r)\nabla r\right)\Delta s \\ &= \phi_q\left(\frac{\delta}{\gamma}\int_{\tau}^{\eta} y(s)\nabla s\right) + \int_{t}^{\sigma(1)} \phi_q\left(\int_{\tau}^{s} y(r)\nabla r\right)\Delta s. \end{aligned} \tag{2.21}$$

Similarly, integrating (2.1) on $(t, \tau)$, we get

$$x(t) = \phi_q\left(\frac{1}{\alpha}\Psi\left(\int_{\xi}^{\tau} y(s)\nabla s\right)\right) + \int_{\rho(0)}^{t} \phi_q\left(\int_{s}^{\tau} y(r)\nabla r\right)\Delta s. \tag{2.22}$$

□

Throughout this paper, we assume that $\tau \in (\rho(0), \sigma(1)) \cap \mathbb{T}$.

**Lemma 2.4.** *Suppose that the conditions in Lemma 2.3 hold. Then there exists a constant $A$ such that the solution $x(t)$ of $p$-Laplacian BVP* (2.1)-(1.2) *satisfies*

$$\max_{t\in[\rho(0),\sigma(1)]} |x(t)| \le A \max_{t\in[\rho(0),\sigma(1)]} \left|x^{\Delta}(t)\right|. \tag{2.23}$$

*Proof.* It is clear that $x(t)$ satisfies

$$\begin{aligned}
x(t) &= x(\rho(0)) + \int_{\rho(0)}^{t} x^{\Delta}(s)\Delta s \\
&= \phi_q\left(\frac{1}{\alpha}\Psi\left(\phi_p\left(x^{\Delta}(\xi)\right)\right)\right) + \int_{\rho(0)}^{t} x^{\Delta}(s)\Delta s \\
&\le \phi_q\left(\frac{1}{\alpha}B_2\phi_p \max_{t\in[\rho(0),\sigma(1)]}\left|x^{\Delta}(t)\right|\right) + \max_{t\in\left[\rho(0),\sigma(1)\right]}\left|x^{\Delta}(t)\right|(t-\rho(0)) \\
&\le \phi_q\left(\frac{B_2}{\alpha}\right)\max_{t\in[\rho(0),\sigma(1)]}\left|x^{\Delta}(t)\right| + \max_{t\in\left[\rho(0),\sigma(1)\right]}\left|x^{\Delta}(t)\right|(\sigma(1)-\rho(0)) \\
&= \left(\phi_q\left(\frac{B_2}{\alpha}\right) + \sigma(1)-\rho(0)\right)\max_{t\in[\rho(0),\sigma(1)]}\left|x^{\Delta}(t)\right|.
\end{aligned} \tag{2.24}$$

Similarly,

$$\begin{aligned}
x(t) &= x(\sigma(1)) - \int_{t}^{\sigma(1)} x^{\Delta}(s)\Delta s \\
&= \phi_q\left(-\frac{\delta}{\gamma}\phi_p\left(x^{\Delta}(\eta)\right)\right) - \int_{t}^{\sigma(1)} x^{\Delta}(s)\Delta s \\
&\le \phi_q\left(\frac{\delta}{\gamma}\right)\max_{t\in[\rho(0),\sigma(1)]}\left|x^{\Delta}(t)\right| + \max_{t\in\left[\rho(0),\sigma(1)\right]}\left|x^{\Delta}(t)\right|(\sigma(1)-t) \\
&\le \phi_q\left(\frac{\delta}{\gamma}\right)\max_{t\in[\rho(0),\sigma(1)]}\left|x^{\Delta}(t)\right| + \max_{t\in\left[\rho(0),\sigma(1)\right]}\left|x^{\Delta}(t)\right|(\sigma(1)-\rho(0)) \\
&= \left(\phi_q\left(\frac{\delta}{\gamma}\right) + \sigma(1)-\rho(0)\right)\max_{t\in[\rho(0),\sigma(1)]}\left|x^{\Delta}(t)\right|.
\end{aligned} \tag{2.25}$$

If we define $A = \min\{\phi_q(B_2/\alpha) + \sigma(1) - \rho(0),\ \phi_q(\delta/\gamma) + \sigma(1) - \rho(0)\}$, we get

$$\max_{t\in[\rho(0),\sigma(1)]}|x(t)| \le A \max_{t\in[\rho(0),\sigma(1)]}\left|x^{\Delta}(t)\right|. \tag{2.26}$$

□

Now, we define a mapping $T : K \to E$ given by

$$(T(x))(t) = \begin{cases} \phi_q\left(\dfrac{1}{\alpha}\Psi\left(\displaystyle\int_{\xi}^{\tau} h(r)f(r,x(r))\nabla r\right)\right) \\ \qquad +\displaystyle\int_{\rho(0)}^{t} \phi_q\left(\int_{s}^{\tau} h(r)f(r,x(r))\nabla r\right)\Delta s, \quad \rho(0) \le t \le \tau; \\ \phi_q\left(\dfrac{\delta}{\gamma}\displaystyle\int_{\tau}^{\eta} h(r)f(r,x(r))\nabla r\right) \\ \qquad +\displaystyle\int_{t}^{\sigma(1)} \phi_q\left(\int_{\tau}^{s} h(r)f(r,x(r))\nabla r\right)\Delta s, \quad \tau < t \le \sigma(1). \end{cases} \tag{2.27}$$

Because of

$$(T(x))^{\Delta}(t) = \begin{cases} \phi_q\left(\displaystyle\int_{t}^{\tau} h(r)f(r,x(r))\nabla r\right), & \rho(0) \le t \le \tau; \\ -\phi_q\left(\displaystyle\int_{\tau}^{t} h(r)f(r,x(r))\nabla r\right), & \tau < t \le \sigma(1), \end{cases} \tag{2.28}$$

we get $(T(x))^{\Delta}(t) \ge 0$, for $t \in [\rho(0),\tau)$ and $(T(x))^{\Delta}(t) \le 0$, for $t \in (\tau,\sigma(1)]$, thus the operator $T$ is monotone increasing on $[\rho(0),\tau)$ and monotone decreasing on $(\tau,\sigma(1)]$ and also $t = \tau$ is the maximum point of the operator $T$. So the operator $T$ is concave on $[0,1]$ and $(T(x))(\tau) = \|T(x)\|$. Therefore, $T(K) \subset K$.

**Lemma 2.5.** *Suppose that the conditions (H1)–(H3) hold. $T : K \to K$ is completely continuous.*

*Proof.* Suppose $P \subset K$ is a bounded set. Let $M > 0$ be such that $\|x\| \le M$, $x \in P$. For any $x \in P$, we have

$$\begin{aligned} \|Tx\| &= (Tx)(\tau) \\ &= \phi_q\left(\frac{1}{\alpha}\Psi\left(\int_{\xi}^{\tau} h(r)f(r,x(r))\nabla r\right)\right) + \int_{\rho(0)}^{\tau} \phi_q\left(\int_{s}^{\tau} h(r)f(r,x(r))\nabla r\right)\Delta s \\ &\le \left\{\phi_q\left(\frac{B_2}{\alpha}\int_{\rho(0)}^{\tau} h(r)\nabla r\right) + \int_{\rho(0)}^{\tau} \phi_q\left(\int_{s}^{\tau} h(r)\nabla r\right)\Delta s\right\}\phi_q\left(\sup_{x\in P, t\in[0,1]} f(t,x(t))\right). \end{aligned} \tag{2.29}$$

Then, $T(P)$ is bounded.

By the Arzela-Ascoli theorem, we can easily see that $T$ is completely continuous operator. □

For convenience, we set

$$R_1 = \frac{2}{L}, \qquad R_2 = \frac{1}{\left(\phi_q(B_2/\alpha) + \sigma(1) - \rho(0)\right)\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)\nabla r\right)}. \tag{2.30}$$

In order to follow the main results of this paper easily, now we state the fixed point theorems which we applied to prove Theorems 3.1–3.4.

**Theorem 2.6** (see [18] (Krasnoselskii fixed point theorem)). *Let $E$ be a Banach space, and let $K \subset E$ be a cone. Assume $\Omega_1$ and $\Omega_2$ are open, bounded subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \longrightarrow K \tag{2.31}$$

*be a completely continuous operator such that either*

(i) $\|Au\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_2$;

(ii) $\|Au\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_1$, $\|Au\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_2$

*hold. Then $A$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

**Theorem 2.7** (see [19] (Schauder fixed point theorem)). *Let $E$ be a Banach space, and let $A : E \to E$ be a completely continuous operator. Assume $K \subset E$ is a bounded, closed, and convex set. If $A(K) \subset K$, then $A$ has a fixed point in $K$.*

**Theorem 2.8** (see [20] (Avery-Henderson fixed point theorem)). *Let $\mathcal{P}$ be a cone in a real Banach space $E$. Set*

$$\mathcal{P}(\phi, r) = \{u \in \mathcal{P} : \phi(u) < r\}. \tag{2.32}$$

*If $\mu$ and $\phi$ are increasing, nonnegative, continuous functionals on $\mathcal{P}$, let $\theta$ be a nonnegative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that for some positive constants $r$ and $M$,*

$$\phi(u) \le \theta(u) \le \mu(u), \qquad \|u\| \le M\phi(u) \tag{2.33}$$

*for all $u \in \overline{\mathcal{P}(\phi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that $\theta(\lambda u) \le \lambda\theta(u)$ for all $0 \le \lambda \le 1$ and $u \in \partial\mathcal{P}(\theta, q)$.*

*If $A : \overline{\mathcal{P}(\phi, r)} \to \mathcal{P}$ is a completely continuous operator satisfying*

(i) $\phi(Au) > r$ *for all* $u \in \partial\mathcal{P}(\phi, r)$,

(ii) $\theta(Au) < q$ *for all* $u \in \partial\mathcal{P}(\theta, q)$,

(iii) $\mathcal{P}(\mu, q) \ne \emptyset$ *and* $\mu(Au) > p$ *for all* $u \in \partial\mathcal{P}(\mu, p)$,

*then $A$ has at least two fixed points $u_1$ and $u_2$ such that*

$$p < \mu(u_1) \quad \text{with } \theta(u_1) < q, \qquad q < \theta(u_2) \quad \text{with } \phi(u_2) < r. \tag{2.34}$$

## 3. Main Results

In this section, we will prove the existence of at least one and two positive solution of $p$-Laplacian BVP (1.1)-(1.2). In the following theorems we will make use of Krasnoselskii, Schauder, and Avery-Henderson fixed point theorems, respectively.

**Theorem 3.1.** *Assume that (H1)–(H3) are satisfied. In addition, suppose that $f$ satisfies*

$(A1)$ $f(t,x) \geq \phi_p(mk_1)$ *for* $\theta k_1/(\sigma(1)-\rho(0)) \leq x \leq k_1$,

$(A2)$ $f(t,x) \leq \phi_p(Mk_2)$ *for* $0 \leq x \leq k_2$,

*where $m \in [R_1,\infty)$ and $M \in (0,R_2]$. Then the $p$-Laplacian BVP* (1.1)-(1.2) *has a positive solution $x(t)$ such that $k_1 \leq \|x\| \leq k_2$.*

*Proof.* Without loss of generality, we suppose $k_1 < k_2$. For any $x \in K$, by Lemma 2.2, we have

$$x(t) \geq \frac{\theta}{\sigma(1)-\rho(0)}\|x\|, \quad \forall t \in [\theta, \sigma(1)-\theta]. \tag{3.1}$$

We define two open subsets $\Omega_1$ and $\Omega_2$ of $E$ such that $\Omega_1 = \{x \in K : \|x\| < k_1\}$ and $\Omega_2 = \{x \in K : \|x\| < k_2\}$.

For $x \in \partial\Omega_1$, by (3.1), we have

$$k_1 = \|x\| \geq x(t) \geq \frac{\theta}{\sigma(1)-\rho(0)}\|x\| \geq \frac{\theta}{\sigma(1)-\rho(0)}k_1, \quad t \in [\theta, \sigma(1)-\theta]. \tag{3.2}$$

For $t \in [\theta, \sigma(1)-\theta]$, if $(A1)$ holds, we will discuss it from three perspectives.

(i) If $\tau \in [\theta, \sigma(1)-\theta]$, thus for $x \in \partial\Omega_1$, by $(A1)$ and Lemma 2.1, we have

$$\begin{aligned}
2\|Tx\| &= 2(Tx)(\tau) \\
&\geq \int_{\rho(0)}^{\tau}\phi_q\left(\int_s^{\tau}h(r)f(r,x(r))\nabla r\right)\Delta s + \int_{\tau}^{\sigma(1)}\phi_q\left(\int_{\tau}^{s}h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\geq mk_1\int_{\theta}^{\tau}\phi_q\left(\int_s^{\tau}h(r)\nabla r\right)\Delta s + mk_1\int_{\tau}^{\sigma(1)-\theta}\phi_q\left(\int_{\tau}^{s}h(r)\nabla r\right)\Delta s \\
&\geq mk_1A(\tau) \geq mk_1L \geq R_1k_1L = 2k_1 = 2\|x\|.
\end{aligned} \tag{3.3}$$

(ii) If $\tau \in [\sigma(1)-\theta, \sigma(1)]$, thus for $x \in \partial\Omega_1$, by $(A1)$ and Lemma 2.1, we have

$$\begin{aligned}
\|Tx\| &= (Tx)(\tau) \\
&\geq \int_{\rho(0)}^{\tau}\phi_q\left(\int_s^{\tau}h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\geq mk_1\int_{\theta}^{\sigma(1)-\theta}\phi_q\left(\int_s^{\sigma(1)-\theta}h(r)\nabla r\right)\Delta s \\
&\geq mk_1A(\sigma(1)-\theta) \geq mk_1L \geq 2k_1 > k_1 = \|x\|.
\end{aligned} \tag{3.4}$$

(iii) If $\tau \in [\rho(0), \theta]$, thus for $x \in \partial\Omega_1$, by $(A1)$ and Lemma 2.1, we have

$$\begin{aligned}
\|Tx\| &= (Tx)(\tau) \\
&\geq \int_{\tau}^{\sigma(1)} \phi_q\left(\int_{\tau}^{s} h(r) f(r, x(r)) \nabla r\right) \Delta s \\
&\geq m k_1 \int_{\theta}^{\sigma(1)-\theta} \phi_q\left(\int_{\theta}^{s} h(r) \nabla r\right) \Delta s \\
&\geq m k_1 A(\theta) \geq m k_1 L \geq 2k_1 > k_1 = \|x\|.
\end{aligned} \tag{3.5}$$

Therefore, we have $\|Tx\| \geq \|x\|$, $\forall x \in \partial\Omega_1$.

On the other hand, as $x \in \partial\Omega_2$, we have $x(t) \leq \|x\| = k_2$, by $(A2)$, we know

$$\begin{aligned}
\|Tx\| &= (Tx)(\tau) \\
&= \phi_q\left(\frac{1}{\alpha}\Psi\left(\int_{\xi}^{\tau} h(r) f(r, x(r)) \nabla r\right)\right) + \int_{\rho(0)}^{\tau} \phi_q\left(\int_{s}^{\tau} h(r) f(r, x(r)) \nabla r\right) \Delta s \\
&\leq \phi_q\left(\frac{B_2}{\alpha}\int_{\rho(0)}^{\sigma(1)} h(r) f(r, x(r)) \nabla r\right) + \int_{\rho(0)}^{\sigma(1)} \phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r) f(r, x(r)) \nabla r\right) \Delta s \\
&\leq M k_2 \left\{\phi_q\left(\frac{B_2}{\alpha}\right) \phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r) \nabla r\right) + \int_{\rho(0)}^{\sigma(1)} \phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r) \nabla r\right) \Delta s\right\} \\
&= M k_2 \left\{\phi_q\left(\frac{B_2}{\alpha}\right) \phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r) \nabla r\right) + (\sigma(1) - \rho(0)) \phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r) \nabla r\right)\right\} \\
&= M k_2 \left(\phi_q\left(\frac{B_2}{\alpha}\right) + \sigma(1) - \rho(0)\right) \phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r) \nabla r\right) \\
&= M k_2 \frac{1}{R_2} < M k_2 \frac{1}{M} = k_2 = \|x\|.
\end{aligned} \tag{3.6}$$

Then, $T$ has a fixed point $x \in (\Omega_2 \setminus \overline{\Omega}_1)$. Obviously, $x$ is a positive solution of the $p$-Laplacian BVP (1.1)-(1.2) and $k_1 \leq \|x\| \leq k_2$. □

Existence of at least one positive solution is also proved using Schauder fixed point theorem (Theorem 2.7). Then we have the following result.

**Theorem 3.2.** *Assume that (H1)–(H3) are satisfied. If $R$ satisfies*

$$\frac{Q}{R_2} \leq R, \tag{3.7}$$

*where $Q$ satisfies*

$$\phi_p(Q) \geq \max_{\|x\|\leq R}\left|f(t,x(t))\right| \quad \textit{for } t\in[0,1], \tag{3.8}$$

*then the p-Laplacian BVP* (1.1)-(1.2) *has at least one positive solution.*

*Proof.* Let $K_R := \{x \in K : \|x\| \leq R\}$. Note that $K_R$ is closed, bounded, and convex subset of $E$ to which the Schauder fixed point theorem is applicable. Define $T : K_R \to E$ as in (2.27) for $t \in [\rho(0), \sigma(1)]$. It can be shown that $T : K_R \to E$ is continuous. Claim that $T : K_R \to K_R$. Let $x \in K_R$. By using the similar methods used in the proof of Theorem 3.1, we have

$$\begin{aligned}
\|Tx\| &= (Tx)(\tau)\\
&= \phi_q\left(\frac{1}{\alpha}\Psi\left(\int_{\xi}^{\tau} h(r)f(r,x(r))\nabla r\right)\right) + \int_{\rho(0)}^{\tau}\phi_q\left(\int_{s}^{\tau} h(r)f(r,x(r))\nabla r\right)\Delta s\\
&\leq \phi_q\left(\frac{B_2}{\alpha}\int_{\rho(0)}^{\sigma(1)} h(r)f(r,x(r))\nabla r\right) + \int_{\rho(0)}^{\sigma(1)}\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)f(r,x(r))\nabla r\right)\Delta s\\
&\leq Q\left(\phi_q\left(\frac{B_2}{\alpha}\right) + \sigma(1) - \rho(0)\right)\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)\nabla r\right)\\
&= Q\frac{1}{R_2} \leq R,
\end{aligned} \tag{3.9}$$

which implies $Tx \in K_R$. The compactness of the operator $T : K_R \to K_R$ follows from the Arzela-Ascoli theorem. Hence $T$ has a fixed point in $K_R$. □

**Corollary 3.3.** *If $f$ is continuous and bounded on* $[0,1]\times\mathbb{R}^+$, *then the p-Laplacian BVP* (1.1)-(1.2) *has a positive solution.*

Now we will give the sufficient conditions to have at least two positive solutions for $p$-Laplacian BVP (1.1)-(1.2). Set

$$P(t) := \phi_q\left(\int_{\theta}^{t} h(r)\nabla r\right) + \phi_q\left(\int_{t}^{\sigma(1)-\theta} h(r)\nabla r\right), \quad t\in[\theta,\sigma(1)-\theta]. \tag{3.10}$$

The function $P(t)$ is positive and continuous on $[\theta,\sigma(1)-\theta]$. Therefore, $P(t)$ has a minimum on $[\theta,\sigma(1)-\theta]$. Hence we suppose there exists $N > 0$ such that $P(t) \geq N$.

Also, we define the nonnegative, increasing continuous functions $\Upsilon, \Phi$, and $\Gamma$ by

$$\begin{aligned}
\Upsilon(x) &= \frac{1}{2}[x(\theta) + x(\sigma(1)-\theta)],\\
\Phi(x) &= \max_{t\in[\rho(0),\theta]\cup[\sigma(1)-\theta,\sigma(1)]} x(t),\\
\Gamma(x) &= \max_{t\in[\rho(0),\sigma(1)]} x(t).
\end{aligned} \tag{3.11}$$

We observe here that, for every $x \in K$, $\Upsilon(x) \leq \Phi(x) \leq \Gamma(x)$ and from Lemma 2.2, $\|x\| \leq ((\sigma(1)-\rho(0))/\theta)\Upsilon(x)$. Also, for $0 \leq \lambda \leq 1$, $\Phi(\lambda x) = \lambda\Phi(x)$.

**Theorem 3.4.** *Assume that (H1)–(H3) are satisfied. Suppose that there exist positive numbers $a < b < c$ such that the function $f$ satisfies the following conditions:*

(i) $f(t,x) \geq \phi_p(ma)$ *for* $x \in [0,a]$,

(ii) $f(t,x) \leq \phi_p(Mb)$ *for* $x \in [0,((\sigma(1)-\rho(0))/\theta)b]$,

(iii) $f(t,x) \geq \phi_p((2/\theta n)c)$ *for* $x \in [(\theta/(\sigma(1)-\rho(0)))c,((\sigma(1)-\rho(0))/\theta)c]$,

*for positive constants $m \in [R_1,\infty)$, $M \in (0,R_2]$, and $n \in (0,N]$. Then the p-Laplacian BVP* (1.1)-(1.2) *has at least two positive solutions $x_1$, $x_2$ such that*

$$\begin{gathered} a < \max_{t\in[\rho(0),\sigma(1)]} x_1(t) \quad \textit{with} \max_{t\in[\rho(0),\theta]\cup[\sigma(1)-\theta,\sigma(1)]} x_1(t) < b, \\ b < \max_{t\in[\rho(0),\theta]\cup[\sigma(1)-\theta,\sigma(1)]} x_2(t) \quad \textit{with} \ \frac{1}{2}[x_2(\theta)+x_2(\sigma(1)-\theta)] < c. \end{gathered} \tag{3.12}$$

*Proof.* Define the cone as in (2.5). From Lemmas 2.2 and 2.3 and the conditions (H1) and (H2), we can obtain $T(K) \subset K$. Also from Lemma 2.5, we see that $T : K \to K$ is completely continuous.

We now show that the conditions of Theorem 2.8 are satisfied.

To fulfill property (i) of Theorem 2.8, we choose $x \in \partial P(\Upsilon,c)$, thus $\Upsilon(x) = (1/2)[x(\theta)+x(\sigma(1)-\theta)] = c$. Recalling that $\|x\| \leq ((\sigma(1)-\rho(0))/\theta)\Upsilon(x) = ((\sigma(1)-\rho(0))/\theta)c$, we have

$$\frac{\theta}{\sigma(1)-\rho(0)}\|x\| \leq x(t) \leq \frac{\sigma(1)-\rho(0)}{\theta}c. \tag{3.13}$$

Then assumption (iii) implies $f(t,x) > \phi_p((2/\theta n)c)$ for $t \in [\theta,\sigma(1)-\theta]$. We have three different cases.

(a) If $\tau \in (\sigma(1)-\theta,\sigma(1))$, we have

$$\begin{aligned} \Upsilon(Tx) &= \frac{1}{2}[Tx(\theta)+Tx(\sigma(1)-\theta)] \\ &\geq Tx(\theta) = \phi_q\left(\frac{1}{\alpha}\Psi\left(\int_\xi^\tau h(r)f(r,x(r))\nabla r\right)\right) + \int_{\rho(0)}^\theta \phi_q\left(\int_s^\tau h(r)f(r,x(r))\nabla r\right)\Delta s \\ &\geq \int_{\rho(0)}^\theta \phi_q\left(\int_s^\tau h(r)f(r,x(r))\nabla r\right)\Delta s \geq \int_{\rho(0)}^\theta \phi_q\left(\int_\theta^{\sigma(1)-\theta} h(r)f(r,x(r))\nabla r\right)\Delta s \\ &\geq \int_{\rho(0)}^\theta \phi_q\left(\int_\theta^{\sigma(1)-\theta} h(r)\phi_p\left(\frac{2}{\theta n}c\right)\nabla r\right)\Delta s = \frac{2}{\theta n}c\phi_q\left(\int_\theta^{\sigma(1)-\theta} h(r)\nabla r\right)(\theta-\rho(0)) \\ &\geq \frac{2}{\theta n}cP(\theta)(\theta) \geq \frac{2}{N}cP(\theta) \geq 2c. \end{aligned} \tag{3.14}$$

Thus we have $\Upsilon(Tx) \geq c$.

(b) If $\tau \in (\rho(0), \theta)$, we have

$$\begin{aligned}
\Upsilon(Tx) &= \frac{1}{2}[Tx(\theta) + Tx(\sigma(1) - \theta)] \geq Tx(\sigma(1) - \theta) \\
&= \phi_q\left(\frac{\delta}{\gamma}\int_{\tau}^{\eta} h(r)f(r,x(r))\nabla r\right) + \int_{\sigma(1)-\theta}^{\sigma(1)} \phi_q\left(\int_{\tau}^{s} h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\geq \int_{\sigma(1)-\theta}^{\sigma(1)} \phi_q\left(\int_{\tau}^{s} h(r)f(r,x(r))\nabla r\right)\Delta s \geq \phi_q\left(\int_{\theta}^{\sigma(1)-\theta} h(r)f(r,x(r))\nabla r\right)(\sigma(1) - \sigma(1) + \theta) \\
&\geq \phi_q\left(\int_{\theta}^{\sigma(1)-\theta} h(r)\phi_p\left(\frac{2}{\theta n}c\right)\nabla r\right)(\theta) \geq \frac{2}{\theta n}c\phi_q\left(\int_{\theta}^{\sigma(1)-\theta} h(r)\nabla r\right)(\theta) \\
&\geq \frac{2}{n}cP(\theta) \geq \frac{2}{N}cP(\theta) \geq 2c.
\end{aligned} \tag{3.15}$$

Thus we have $\Upsilon(Tx) \geq c$.

(c) If $\tau \in [\theta, \sigma(1) - \theta]$, we have

$$\begin{aligned}
2\Upsilon(Tx) &= Tx(\theta) + Tx(\sigma(1) - \theta) \\
&\geq \int_{\rho(0)}^{\theta} \phi_q\left(\int_{s}^{\tau} h(r)f(r,x(r))\nabla r\right)\Delta s + \int_{\sigma(1)-\theta}^{\sigma(1)} \phi_q\left(\int_{\tau}^{s} h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\geq \frac{2}{\theta n}c\left\{\int_{\rho(0)}^{\theta} \phi_q\left(\int_{\theta}^{\tau} h(r)\nabla r\right)\Delta s + \int_{\sigma(1)-\theta}^{\sigma(1)} \phi_q\left(\int_{\tau}^{\sigma(1)-\theta} h(r)\nabla r\right)\Delta s\right\} \\
&\geq \frac{2}{\theta n}c\left\{\phi_q\left(\int_{\theta}^{\tau} h(r)\nabla r\right)(\theta - \rho(0)) + \phi_q\left(\int_{\tau}^{\sigma(1)-\theta} h(r)\nabla r\right)(\theta)\right\} \\
&\geq \frac{2}{\theta n}c\left\{\phi_q\left(\int_{\theta}^{\tau} h(r)\nabla r\right) + \phi_q\left(\int_{\tau}^{\sigma(1)-\theta} h(r)\nabla r\right)\right\}(\theta) \\
&\geq \frac{2}{n}cP(\tau) \geq \frac{2}{N}cN = 2c.
\end{aligned} \tag{3.16}$$

Thus we have $\Upsilon(Tx) \geq c$ and condition (i) of Theorem 2.8 holds. Next we will show condition (ii) of Theorem 2.8 is satisfied. If $x \in \partial\mathcal{P}(\Phi, b)$, then $\max_{t\in[\rho(0),\theta]\cup[\sigma(1)-\theta,\sigma(1)]} x(t) = b$.

Noting that

$$\|x\| \leq \frac{\sigma(1) - \rho(0)}{\theta}\Upsilon(x) \leq \frac{\sigma(1) - \rho(0)}{\theta}\Phi(x) = \frac{\sigma(1) - \rho(0)}{\theta}b, \tag{3.17}$$

we have $0 \leq x(t) \leq ((\sigma(1) - \rho(0))/\theta)b$, for $t \in [\rho(0), \sigma(1)]$.

Then (ii) yields $f(t,x) \le \phi_p(Mb)$ for $t \in [\rho(0), \sigma(1)]$.

As $Tx \in K$, so

$$\begin{aligned}
\Phi(Tx) &= \max_{t\in[\rho(0),\theta]\cup[\sigma(1)-\theta,\sigma(1)]} Tx(t) \le Tx(\tau) \\
&= \phi_q\left(\frac{1}{\alpha}\Psi\left(\int_{\xi}^{\tau} h(r)f(r,x(r))\nabla r\right)\right) + \int_{\rho(0)}^{\tau}\phi_q\left(\int_{s}^{\tau} h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\le \phi_q\left(\frac{B_2}{\alpha}\int_{\rho(0)}^{\sigma(1)} h(r)f(r,x(r))\nabla r\right) + \int_{\rho(0)}^{\sigma(1)}\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\le \phi_q\left(\frac{B_2}{\alpha}\right)\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)\phi_p(bM)\nabla r\right) + \int_{\rho(0)}^{\sigma(1)}\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)\phi_p(bM)\nabla r\right)\Delta s \\
&= bM\phi_q\left(\frac{B_2}{\alpha}\right)\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)\nabla r\right) + bM\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)\nabla r\right)(\sigma(1)-\rho(0)) \\
&= bM\phi_q\left(\int_{\rho(0)}^{\sigma(1)} h(r)\nabla r\right)\left(\phi_q\left(\frac{B_2}{\alpha}\right) + \sigma(1)-\rho(0)\right) \le bR_2\frac{1}{R_2} = b.
\end{aligned} \tag{3.18}$$

So condition (ii) of Theorem 2.8 holds.

To fulfill property (iii) of Theorem 2.8, we note $x_*(t) = a/2$, $t \in [\rho(0), \sigma(1)]$ is a member of $\mathcal{P}(\Gamma, a)$ and $\Gamma(x_*) = a/2$, so $\mathcal{P}(\Gamma, a) \neq \emptyset$. Now choose $x \in \partial\mathcal{P}(\Gamma, a)$, then $\Gamma(x) = \max_{t\in[\rho(0),\sigma(1)]} x(t) = a$ and this implies that $0 \le x(t) \le a$ for $t \in [\rho(0), \sigma(1)]$. It follows from the assumption (i), we have $f(t,x) \ge \phi_p(ma)$ for $t \in [\rho(0), \sigma(1)]$. As before we obtain the following cases.

(a) If $\tau < \theta$, we have

$$\begin{aligned}
\Gamma(Tx) &= \max_{t\in[\rho(0),\sigma(1)]} Tx(t) = Tx(\tau) \\
&\ge \int_{\tau}^{\sigma(1)}\phi_q\left(\int_{\tau}^{s} h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\ge \int_{\theta}^{\sigma(1)-\theta}\phi_q\left(\int_{\tau}^{s} h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\ge \int_{\theta}^{\sigma(1)-\theta}\phi_q\left(\int_{\theta}^{s} h(r)f(r,x(r))\nabla r\right)\Delta s \\
&\ge \int_{\theta}^{\sigma(1)-\theta}\phi_q\left(\int_{\theta}^{s} h(r)\phi_p(ma)\nabla r\right)\Delta s \\
&= ma\int_{\theta}^{\sigma(1)-\theta}\phi_q\left(\int_{\theta}^{s} h(r)\nabla r\right)\Delta s \\
&= maA(\theta) \ge R_1 aL = 2a \ge a.
\end{aligned} \tag{3.19}$$

Thus we have $\Gamma(Tx) \ge a$.

(b) If $\tau \in [\theta, \sigma(1) - \theta]$, we have

$$\begin{aligned}
2\Gamma(Tx) &= 2Tx(\tau) \\
&\geq \int_{\rho(0)}^{\tau} \phi_q\left(\int_s^{\tau} h(r) f(r, x(r)) \nabla r\right) \Delta s + \int_{\tau}^{\sigma(1)} \phi_q\left(\int_{\tau}^{s} h(r) f(r, x(r)) \nabla r\right) \Delta s \\
&\geq \int_{\theta}^{\tau} \phi_q\left(\int_s^{\tau} h(r) \phi_p(ma) \nabla r\right) \Delta s + \int_{\tau}^{\sigma(1)-\theta} \phi_q\left(\int_{\tau}^{s} h(r) \phi_p(ma) \nabla r\right) \Delta s \\
&\geq ma\left\{\int_{\theta}^{\tau} \phi_q\left(\int_s^{\tau} h(r) \nabla r\right) \Delta s + \int_{\tau}^{\sigma(1)-\theta} \phi_q\left(\int_{\tau}^{s} h(r) \nabla r\right) \Delta s\right\} \\
&= maA(\tau) \geq R_1 aL \geq a.
\end{aligned} \tag{3.20}$$

Thus we have $\Gamma(Tx) \geq a$.

(c) If $\tau > \sigma(1) - \theta$, we have

$$\begin{aligned}
\Gamma(Tx) &= Tx(\tau) \\
&\geq \int_{\rho(0)}^{\tau} \phi_q\left(\int_s^{\tau} h(r) f(r, x(r)) \nabla r\right) \Delta s \\
&\geq \int_{\rho(0)}^{\tau} \phi_q\left(\int_s^{\tau} h(r) \phi_p(ma) \nabla r\right) \Delta s \\
&\geq ma\left\{\int_{\theta}^{\sigma(1)-\theta} \phi_q\left(\int_s^{\sigma(1)-\theta} h(r) \nabla r\right) \Delta s\right\} \\
&= maA(\sigma(1) - \theta) \geq R_1 aL \geq a.
\end{aligned} \tag{3.21}$$

Thus we have $\Gamma(Tx) \geq a$.

Therefore, condition (iii) of Theorem 2.8 holds. Since all conditions of Theorem 2.8 are satisfied, the $p$-Laplacian BVP (1.1)-(1.2) has at least two positive solutions $x_1$, $x_2$ such that

$$\begin{gathered}
a < \max_{t \in [\rho(0), \sigma(1)]} x_1(t) \quad \text{with} \max_{t \in [\rho(0), \theta] \cup [\sigma(1)-\theta, \sigma(1)]} x_1(t) < b, \\
b < \max_{t \in [\rho(0), \theta] \cup [\sigma(1)-\theta, \sigma(1)]} x_2(t) \quad \text{with} \quad \frac{1}{2}[x_2(\theta) + x_2(\sigma(1) - \theta)] < c.
\end{gathered} \tag{3.22}$$

□

## 4. Monotone Method

In this section, we will prove the existence of solution of $p$-Laplacian BVP (1.1)-(1.2) by using upper and lower solution method. We define the set

$$D := \left\{ x : \left(\phi_p\left(x^{\Delta}\right)\right)^{\nabla} \text{ is continuous on } [0,1] \right\}. \tag{4.1}$$

*Definition 4.1.* A real-valued function $u(t) \in D$ on $[\rho(0), \sigma(1)]$ is a lower solution for (1.1)-(1.2) if

$$\begin{gathered}\left(\phi_p\left(u^{\Delta}\right)\right)^{\nabla}(t) + h(t) f(t, u(t)) > 0, \quad t \in [0,1], \\ \alpha\phi_p(u(\rho(0)) - \Psi\left(\phi_p\left(u^{\Delta}(\xi)\right) \le 0, \qquad \gamma\phi_p(u(\sigma(1))) + \delta\phi_p(u^{\Delta}(\eta) \le 0.\end{gathered} \tag{4.2}$$

Similarly, a real-valued function $v(t) \in D$ on $[\rho(0), \sigma(1)]$ is an upper solution for (1.1)-(1.2) if

$$\begin{gathered}\left(\phi_p\left(v^{\Delta}\right)\right)^{\nabla}(t) + h(t) f(t, v(t)) < 0, \quad t \in [0,1], \\ \alpha\phi_p(v(\rho(0)) - \Psi\left(\phi_p\left(v^{\Delta}(\xi)\right) \ge 0, \qquad \gamma\phi_p(v(\sigma(1))) + \delta\phi_p(v^{\Delta}(\eta) \ge 0.\end{gathered} \tag{4.3}$$

We will prove when the lower and the upper solutions are given in the well order, that is, $u \le v$, the $p$-Laplacian BVP (1.1)-(1.2) admits a solution lying between both functions.

**Theorem 4.2.** *Assume that (H1)–(H3) are satisfied and $u$ and $v$ are, respectively, lower and upper solutions for the $p$-Laplacian BVP* (1.1)-(1.2) *such that $u \le v$ on $[\rho(0), \sigma(1)]$. Then the $p$-Laplacian BVP* (1.1)-(1.2) *has a solution $x(t) \in [u(t), v(t)]$ on $[\rho(0), \sigma(1)]$.*

*Proof.* Consider the $p$-Laplacian BVP:

$$\begin{gathered}\left(\phi_p\left(x^{\Delta}\right)\right)^{\nabla}(t) + h(t) F(t, x(t)) = 0, \quad t \in [0,1], \\ \alpha\phi_p(x(\rho(0)) - \Psi\left(\phi_p\left(x^{\Delta}(\xi)\right) = 0, \qquad \gamma\phi_p(x(\sigma(1))) + \delta\phi_p(x^{\Delta}(\eta) = 0,\end{gathered} \tag{4.4}$$

where

$$F(t, x(t)) = \begin{cases} f(t, v(t)), & x(t) > v(t), \\ f(t, x(t)), & u(t) \le x(t) \le v(t), \\ f(t, u(t)), & x(t) < u(t), \end{cases} \tag{4.5}$$

for $t \in [0,1]$

Clearly, the function $F$ is bounded for $t \in [0,1]$ and satisfies condition (H1). Thus by Theorem 3.2, there exists a solution $x(t)$ of the $p$-Laplacian BVP (4.4). We first show that $x(t) \le v(t)$ on $[\rho(0), \sigma(1)]$. Set $z(t) = x(t) - v(t)$. If $x(t) \le v(t)$ on $[\rho(0), \sigma(1)]$ is not true, then there exists a $t_0 \in [\rho(0), \sigma(1)]$ such that $z(t_0) = \max_{t\in[\rho(0),\sigma(1)]}\{x(t) - v(t)\} > 0$ has a positive maximum. Consequently, we know that $z^{\Delta}(t_0) \le 0$ and there exists $t_1 \in (\rho(0), t_0)$ such that $z^{\Delta}(t) \ge 0$ on $[t_1, t_0)$. On the other hand by the continuity of $z(t)$ at $t_0$, we know there exists $t_2 \in (\rho(0), t_0)$ such that $z(t) > 0$ on $[t_2, t_0]$. Let $\bar{t} = \max\{t_1, t_2\}$, then we have $z^{\Delta}(t) \ge 0$ on $[\bar{t}, t_0)$. Thus we get

$$
\begin{aligned}
z^{\Delta}\left(\bar{t}\right) \ge 0 &\Longrightarrow \phi_p\left(x^{\Delta}\left(\bar{t}\right)\right) \ge \phi_p\left(v^{\Delta}\left(\bar{t}\right)\right), \\
z^{\Delta}(t_0) \le 0 &\Longrightarrow \phi_p\left(x^{\Delta}(t_0)\right) \le \phi_p\left(v^{\Delta}(t_0)\right).
\end{aligned}
\tag{4.6}
$$

Therefore,

$$
\begin{aligned}
0 &\ge \left[\phi_p\left(x^{\Delta}(t_0)\right) - \phi_p\left(v^{\Delta}(t_0)\right)\right] - \left[\phi_p\left(x^{\Delta}\left(\bar{t}\right)\right) - \phi_p\left(\left(v^{\Delta}\right)\left(\bar{t}\right)\right)\right] \\
&= \int_{\bar{t}}^{t_0} \left[\phi_p\left(x^{\Delta}\right) - \phi_p\left(v^{\Delta}\right)\right]^{\nabla}(t)\nabla t \\
&= \int_{\bar{t}}^{t_0} \left[\left(\phi_p\left(x^{\Delta}\right)\right)^{\nabla}(t) - \left(\phi_p\left(v^{\Delta}\right)\right)^{\nabla}(t)\right]\nabla t \\
&> \int_{\bar{t}}^{t_0} \left[-h(t)f(t, v(t)) + h(t)f(t, v(t))\right]\nabla t = 0,
\end{aligned}
\tag{4.7}
$$

which is a contradiction and thus $t_0$ cannot be an element of $(\rho(0), \sigma(1))$.

If $t_0 = \rho(0)$, from the boundary conditions, we have

$$
\begin{aligned}
\alpha\phi_p(x(\rho(0))) \le B_2\phi_p\left(x^{\Delta}(\xi)\right) &\Longrightarrow \phi_p\left(\alpha^{1-q}x(\rho(0))\right) \le \phi_p\left(B_2^{1-q}x^{\Delta}(\xi)\right), \\
\alpha\phi_p(v(\rho(0))) \ge B_1\phi_p\left(v^{\Delta}(\xi)\right) &\Longrightarrow \phi_p\left(\alpha^{1-q}v(\rho(0))\right) \ge \phi_p\left(B_1^{1-q}v^{\Delta}(\xi)\right).
\end{aligned}
\tag{4.8}
$$

Thus we get

$$
\alpha^{1-q}x(\rho(0)) \le B_2^{1-q}x^{\Delta}(\xi), \qquad \alpha^{1-q}v(\rho(0)) \ge B_1^{1-q}v^{\Delta}(\xi).
\tag{4.9}
$$

From this inequalities, we have

$$
\begin{gathered}
\alpha^{1-q}(x - v)(\rho(0)) \le B_2^{1-q}x^{\Delta}(\xi) - B_1^{1-q}v^{\Delta}(\xi) \le B_1^{1-q}(x - v)^{\Delta}(\xi), \\
\alpha^{1-q}z(\rho(0)) \le B_1^{1-q}z^{\Delta}(\xi) \le 0,
\end{gathered}
\tag{4.10}
$$

which is a contradiction.

If $t_0 = \sigma(1)$, from the boundary conditions, we have

$$\begin{aligned}\gamma\phi_p(x(\sigma(1))) &= -\delta\phi_p\left(x^\Delta(\eta)\right) \Longrightarrow \phi_p\left(\gamma^{1-q}x(\sigma(1))\right)\\&= -\phi_p\left(\delta^{1-q}x^\Delta(\eta)\right) = \phi_p\left(-\delta^{1-q}x^\Delta(\eta)\right),\\\gamma\phi_p(v(\sigma(1))) &\geq -\delta\phi_p\left(v^\Delta(\eta)\right) \Longrightarrow \phi_p\left(\gamma^{1-q}v(\sigma(1))\right)\\&\geq -\phi_p\left(\delta^{1-q}v^\Delta(\eta)\right) = \phi_p\left(-\delta^{1-q}v^\Delta(\eta)\right).\end{aligned} \tag{4.11}$$

Thus we get

$$\gamma^{1-q}x(\sigma(1)) = \delta^{1-q}x^\Delta(\eta), \qquad \gamma^{1-q}v(\sigma(1)) \geq -\delta^{1-q}v^\Delta(\eta). \tag{4.12}$$

From this inequalities, we have

$$\begin{aligned}\gamma^{1-q}(x-v)(\sigma(1)) &\leq -\delta^{1-q}(x-v)^\Delta(\eta),\\\gamma^{1-q}z(\sigma(1)) &\leq -\delta^{1-q}z^\Delta(\eta) \leq 0,\end{aligned} \tag{4.13}$$

which is a contradiction. Thus we have $x(t) \leq v(t)$ on $[\rho(0), \sigma(1)]$.

Similarly, we can get $u(t) \leq x(t)$ on $[\rho(0), \sigma(1)]$. Thus $x(t)$ is a solution of $p$-Laplacian BVP (1.1)-(1.2) which lies between $u$ and $v$. □

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